

Turbulent Flows
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Solution to Exercise 10.9

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- a) Consider the log-law region of a wall-bounded turbulent flow. The turbulent viscosity hypothesis says

$$-\langle uv \rangle = \nu_T \frac{\partial \langle U \rangle}{\partial y}. \quad (1)$$

With $\nu_T = C_\mu k^2 / \varepsilon$, we get

$$\langle uv \rangle = -C_\mu k^2 / \varepsilon \frac{\partial \langle U \rangle}{\partial y}. \quad (2)$$

According to the log-law

$$u^+ = \frac{1}{\kappa} \ln y^+ + B \quad (3)$$

where $y^+ = \frac{y}{\delta_\nu} = \frac{u_\tau y}{\nu}$, $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$, $\delta_\nu = \frac{\nu}{u_\tau}$ and $u^+ = \frac{\langle U \rangle}{u_\tau}$, we get

$$\begin{aligned} \frac{\partial \langle U \rangle}{\partial y} &= \frac{u_\tau^2}{\nu} \frac{du^+}{dy^+} = \frac{u_\tau^2}{\nu \kappa y^+} \\ &= \frac{u_\tau}{\kappa y}. \end{aligned} \quad (4)$$

Substituting Eq. 4 into Eq. 2, we get

$$\langle uv \rangle = -C_\mu k^2 u_\tau / (\kappa \varepsilon y). \quad (5)$$

In the log-law region of a wall-bounded turbulent flow, $\langle uv \rangle \approx -\tau_w / \rho$, so

$$\langle uv \rangle \approx -u_\tau^2. \quad (6)$$

Substituting Eq. 6 into Eq. 5, we get

$$\varepsilon = \frac{C_\mu k^2}{u_\tau \kappa y}. \quad (7)$$

b) Given $\mathcal{P} = \varepsilon$ and $\mathcal{P} = -\langle uv \rangle \frac{d\langle U \rangle}{dt}$, we get

$$\langle uv \rangle \frac{\partial \langle U \rangle}{\partial y} = -\frac{C_\mu k^2}{u_\tau \kappa y}. \quad (8)$$

Substituting Eqs. 4 and 6 into Eq. 8, we get

$$C_\mu^{\frac{1}{2}} k = u_\tau^2. \quad (9)$$

So substituting Eq. 9 into Eq. 7, we get

$$\varepsilon = \frac{C_\mu k^2}{u_\tau \kappa y} = \frac{C_\mu^{\frac{1}{2}} k u_\tau^2}{u_\tau \kappa y} = \frac{C_\mu^{\frac{1}{2}} k u_\tau}{\kappa y}. \quad (10)$$

c) Multiplying Eq. 10 with Eq. 7 and then taking the square root, we get

$$\varepsilon = \left(\frac{C_\mu^{\frac{1}{2}} k u_\tau}{\kappa y} \frac{C_\mu k^2}{u_\tau \kappa y} \right)^{\frac{1}{2}} = \frac{C_\mu^{\frac{3}{4}} k^{\frac{3}{2}}}{\kappa y}. \quad (11)$$

Comparing it with $\varepsilon = c^3 k^{\frac{3}{2}} / \ell_m$ (Eq.10.43), we get $C_\mu = c^4$ and $\ell_m = \kappa y$.

d) Equation (10.70) says

$$0 = \frac{d}{dy} \left(\frac{\nu_T}{\sigma_\varepsilon} \frac{d\varepsilon}{dy} \right) + C_{\varepsilon 1} \frac{\mathcal{P}\varepsilon}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k}. \quad (12)$$

Substituting Eq. 11 into Eq. 12 and with $\mathcal{P} = \varepsilon$, we get

$$\begin{aligned} 0 &= - \frac{d}{dy} \left(\frac{C_\mu k^2}{\sigma_\varepsilon y} \right) + C_{\varepsilon 1} \frac{\mathcal{P}\varepsilon}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} \\ &= \frac{C_\mu k^2}{\sigma_\varepsilon y^2} + C_{\varepsilon 1} \frac{\mathcal{P}\varepsilon}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} \\ &= \frac{C_\mu k^2}{\sigma_\varepsilon y^2} + (C_{\varepsilon 1} - C_{\varepsilon 2}) \frac{C_\mu^{\frac{3}{2}} k^2}{\kappa^2 y^2} \end{aligned} \quad (13)$$

Finally, we get

$$\kappa^2 = \sigma_\varepsilon C_\mu^{\frac{1}{2}} (C_{\varepsilon 2} - C_{\varepsilon 1}). \quad (14)$$

e) Using Eq. 11, the lengthscale

$$L = \frac{k^{\frac{3}{2}}}{\varepsilon} = \frac{k^{\frac{3}{2}}}{\frac{C_\mu^{\frac{3}{4}} k^{\frac{3}{2}}}{\kappa y}} = C_\mu^{-\frac{3}{4}} \kappa y. \quad (15)$$

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