Turbulent Flows
Stephen B. Pope
Cambridge University Press (2000)

Solution to Exercise 10.9

Prepared by: Zhuyin Ren *Date:* 3/26/03

a) Consider the log-law region of a wall-bounded turbulent flow. The turbulent viscosity hypothesis say

$$-\langle uv\rangle = \nu_T \frac{\partial \langle U\rangle}{\partial y}. \tag{1}$$

With $\nu_T = C_\mu k^2/\varepsilon$, we get

$$\langle uv\rangle = -C_\mu k^2/\varepsilon \frac{\partial \langle U\rangle}{\partial y}. \tag{2}$$

According to the log-law

$$u^+ = \frac{1}{\kappa}\ln y^+ + B \tag{3}$$

where $y^+ = \frac{y}{\delta_\nu} = \frac{u_\tau y}{\nu}$, $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$, $\delta_\nu = \frac{\nu}{u_\tau}$ and $u^+ = \frac{\langle U\rangle}{u_\tau}$, we get

$$\begin{aligned} \frac{\partial \langle U\rangle}{\partial y} &= \frac{{u_\tau}^2}{\nu}\frac{du^+}{dy^+} = \frac{{u_\tau}^2}{\nu\kappa y^+} \\ &= \frac{u_\tau}{\kappa y}. \end{aligned} \tag{4}$$

Substituting Eq. 4 into Eq. 2, we get

$$\langle uv\rangle = -C_\mu k^2 u_\tau/(\kappa\varepsilon y). \tag{5}$$

In the log-law region of a wall-bounded turbulent flow, $\langle uv\rangle \approx -\tau_w/\rho$, so

$$\langle uv\rangle \approx -{u_\tau}^2. \tag{6}$$

Substituting Eq. 6 into Eq. 5, we get

$$\varepsilon = \frac{C_\mu k^2}{u_\tau \kappa y}. \tag{7}$$

b) Given $\mathcal{P} = \varepsilon$ and $\mathcal{P} = -\langle uv \rangle \frac{d\langle U \rangle}{dt}$, we get

$$\langle uv \rangle \frac{\partial \langle U \rangle}{\partial y} = -\frac{C_\mu k^2}{u_\tau \kappa y}. \tag{8}$$

Substituting Eqs. 4 and 6 into Eq. 8, we get

$$C_\mu{}^{\frac{1}{2}} k = u_\tau{}^2. \tag{9}$$

So substituting Eq. 9 into Eq. 7, we get

$$\varepsilon = \frac{C_\mu k^2}{u_\tau \kappa y} = \frac{C_\mu{}^{\frac{1}{2}} k u_\tau{}^2}{u_\tau \kappa y} = \frac{C_\mu{}^{\frac{1}{2}} k u_\tau}{\kappa y}. \tag{10}$$

c) Multiplying Eq. 10 with Eq. 7 and then taking the square root, we get

$$\varepsilon = \left(\frac{C_\mu{}^{\frac{1}{2}} k u_\tau}{\kappa y} \frac{C_\mu k^2}{u_\tau \kappa y} \right)^{\frac{1}{2}} = \frac{C_\mu{}^{\frac{3}{4}} k^{\frac{3}{2}}}{\kappa y}. \tag{11}$$

Comparing it with $\varepsilon = c^3 k^{\frac{3}{2}} / \ell_m$ (Eq.10.43), we get $C_\mu = c^4$ and $\ell_m = \kappa y$.

d) Equation (10.70) says

$$0 = \frac{d}{dy} \left(\frac{\nu_T}{\sigma_\varepsilon} \frac{d\varepsilon}{dy} \right) + C_{\varepsilon 1} \frac{\mathcal{P}\varepsilon}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k}. \tag{12}$$

Substituting Eq. 11 into Eq. 12 and with $\mathcal{P} = \varepsilon$, we get

$$\begin{aligned} 0 &= - \frac{d}{dy} \left(\frac{C_\mu k^2}{\sigma_\varepsilon y} \right) + C_{\varepsilon 1} \frac{\mathcal{P}\varepsilon}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} \\ &= \frac{C_\mu k^2}{\sigma_\varepsilon y^2} + C_{\varepsilon 1} \frac{\mathcal{P}\varepsilon}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} \\ &= \frac{C_\mu k^2}{\sigma_\varepsilon y^2} + (C_{\varepsilon 1} - C_{\varepsilon 2}) \frac{C_\mu{}^{\frac{3}{2}} k^2}{\kappa^2 y^2} \end{aligned} \tag{13}$$

Finally, we get

$$\kappa^2 = \sigma_\varepsilon {C_\mu}^{\frac{1}{2}} \left(C_{\varepsilon 2} - C_{\varepsilon 1} \right). \tag{14}$$

e) Using Eq. 11, the lengthscale

$$L = \frac{k^{\frac{3}{2}}}{\varepsilon} = \frac{k^{\frac{3}{2}}}{\frac{{C_\mu}^{\frac{3}{4}} k^{\frac{3}{2}}}{\kappa y}} = {C_\mu}^{-\frac{3}{4}} \kappa y. \tag{15}$$